J. Social Biol. Struct. 1989 **12**, 73–81

McClain's circle and Plato's harp

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McClain depicted Plato's city of Magnesia as a circle and found that its structure was based on mathematical ratios which coincide with Just Intonation set of musical intervals (including non-standard intervals 4/7 and 7/8). Later McClain noted that the same set of intervals was deduced formally by the author of this paper from his model of human reflexion.

In this work the author puts forth the hypothesis that Plato might have known of some formal correlations analogous to those used in this model.

(. . .) the ancient cosmos is a malleable sculptured unity, like some big figure or statue or even an exquisitely tuned instrument that generates specific sounds Losev (1963, p. 229).

Introduction

The following analysis was stimulted by McClain's 'Comments' (McClain, 1987) on my work on human reflexion (Lefebvre, 1987). I tried to demonstrate how a set of musical intervals known as Just Intonation (a 'natural' scale) can be deduced from simple assumptions about the structure and function of inner reflexional processes. The peculiarity of this result is that in addition to generating intervals with the tempered approximations used broadly today, the set I deduced contains two additional intervals, 4/7 and 7/8. These have been used extremely rarely in European music since the tempered scale was introduced.

McClain mentioned in his 'Comments' that he found the same set of musical intervals (including 4/7 and 7/8) when he used his own system of Pythagorean musicology to reconstruct Plato's model of the city of Magnesia, as described in *The Laws*. This astonishing coincidence indicated that Plato may have used a formalism analogous to the one I used.

We know that during the last years of his life, Plato became a Pythagorean. In the opinion of many scholars, Plato described his views of that period in 'Lecture on the Good', which unfortunately, has not reached us. Questions about the character of these lectures have been hotly debated for many centuries. Among recent authors, some believe there was an 'esoteric teaching' in the Academy, and that the content of 'Plato's Lecture on the Good' was transmitted only orally (Findley, 1974; Gaiser, 1980). On the other hand, Sayre (1983) hypothesized that Plato's late ideas are described in his *Philebus*. Today, however, little doubt remains that Plato's late ideas were permeated with mathematics, even though the relationships between mathematics and Plato's

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conceptual system remains unclear. Moreover, we may suppose that toward the end of his life Plato succeeded in constructing a formal model of *Universum*. *The Laws* were written during this period. That is, we hypothesize that the intervals found by McClain in the structure of Magnesia were deduced by Plato using a formal model of *Universum*.

The first step is to attempt to reconstruct Plato's formal *Universum* by assuming that he had discovered some of the relations present in a formal model of reflexion, or, more precisely, that he knew the relations which predetermine the uniqueness of the Just Intonation Set.

A model of reflexion

For the last decade I have worked on developing a formal model of human reflexion (Lefebvre, 1977, 1982, 1985, 1987). The essence of this model consists of a theoretical representation of a human being whose inner domain contains 'natural' and 'cognizant' images of the self and of others. This representation permits constructing theoretical analogies of 'higher' feelings in the area which is usually associated with self-reflexive impressions such as 'conscience': guilt, condemnation, anguish, inner pain, etc. In addition to these proximate feelings, the model also contains analogies of the cognizant images or these feelings; that is sensations of having feelings. The first to pay attention to this level of reflexional phenomena was the Prague psychiatrist Max Levi (1908). However, the rapid development of psychoanalysis overshadowed this important discovery. Sixty years later the Soviet psychiatrist, V. I. Reznik (1969), analysed Levi's contribution in detail.[†]

My reflexional model allowed us also to find the analogies for the intensities of inner feelings (called 'deep') and for feelings of feelings (called 'surface feelings'). I put forth the hypothesis that musical intervals influence the intensities of both deep and surface feelings. A surface feeling can be considered a special model of a deep feeling, which is 'inserted' into a subject. I supposed that the most attractive intervals for a subject are those for which the ratio of the intensity of a feeling-original (deep feeling) to that of a feeling-model (surface feeling) is an integer. The model dictates that the ratio of feelings is an integer if and only if the intervals look as follows:

$$\frac{k+1}{k+2} \quad \text{or} \quad \frac{k+1}{2k+1}, \quad k = 1, 2, 3, \ldots$$

I called these intervals *elite intervals* (Lefebvre, 1987). It turns out that the interval 2/3 plays a special role. This interval generates the ratio of feelings equal to 1: the intensity of a feeling-original and that of a feeling-model are equal to each other.

It was very natural to assume next that a subject's feelings are influenced not only by intervals but also by the ratio of intervals. From all of the elite intervals, I singled out those which together with the interval 2/3 generate integer ratios of the intensities of deep and surface feelings. The set of these can be found to be the solution of two Diophantine equations. Each interval from this set generates an integer ratio of inner feelings by itself and together with the interval 2/3. If we add the interval 2/3 to this set, we obtain the intervals which I labeled 'super-elite'. Below you can see all of the super-elite intervals

[†] Reznik's paper published in the proceedings of a conference forgotten today, has remained almost unknown to the scientific community. I ran into it accidentally while looking through some old books.

plus their octave complements (which are not super-elite intervals).

Super-elite intervals:	$\frac{8}{15}$	$\frac{5}{9}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{8}{9}$
Octave complements:	$\frac{15}{16}$	$\frac{9}{10}$	$\frac{7}{8}$				$\frac{5}{8}$		$\frac{9}{16}$

We see that this set coincides with the Just Intonation set which includes the two non-standard intervals 4/7 and 7/8.

McClain's comments

McClain writes:

The total set of musical ratios which Lefebvre discovers in his model have the number 5040 as their least common denominator (test it yourself). This is the exact number of citizens in Plato's model city of Magnesia, ruled by 37 guardians educated in music and mathematics. Figure 1 shows Plato's 37 guardians in a tone circle, a cyclic logarthmic scale on base 2. Lefebvre's 12 standard

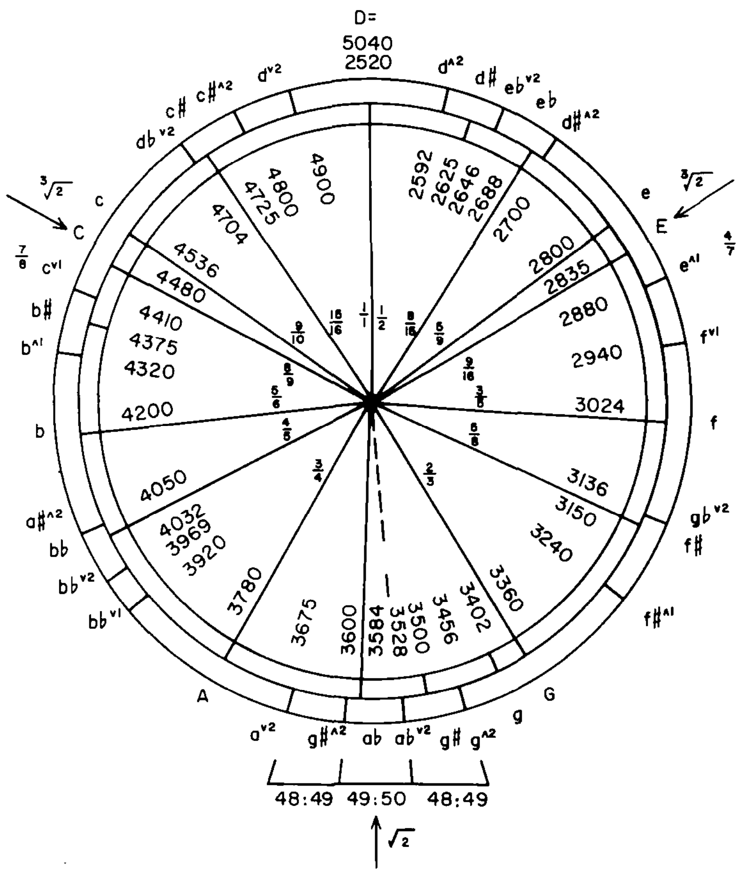

Fig. 1. 'The 37 guardians of Magnesia. Plato's gardians from The Laws *approximate a 36-tone equal temperament as closely as possible within the limits of factorial 7 (i.e. 7!) = 5040 . . .' (McClain, 1987). I call this construction* McClains Circle *and I have added numerical ratios in the central part of the drawing and two fractions on the outer rim.*

'just' ratios, read either direction around Plato's circle, divide the circular territory of Magnesia into 12 radial sections, 'as nearly equal as possible' with natural numbers up to 5040. The sectional subdivisions in the outer rim are those quarter tones and third tones already needed by Greek theorists in the fourth centruy BC; they are generated by the 'septimal' values of 4/7 and 7/8 associated with the 'natural seventh.' All these 'boundary markers' are visited by future guardians of Plato's state during their two years of required military training, circling clockwise one year and counterclockwise the next (McClain, 1976). Lefebvre cannot be expected to have recognized Plato's model in his own (*The Laws* is yet to be studied by classicists familiar with its musical content, accessible only in the last 10 years.) One can visualize a 36-tone, equal-division octave superimposed on Plato's map, or, with a little digital industry, calculate precise differences by the formulae given earlier. (Why Plato's circle has an extra member has been explained in detail in my book. [*McClain, 1978—V.L.*]) Lefebvre's numerical postulates and gamma algebra have coerced us inexorably into Plato's training camp for 'guardians of the state.' It is no wonder that we feel ourselves in the grip of a superior power (McClain, 1987, pp. 210–211).

I am no less astonished by this coincidence than McClain. This paper can be regarded at as an attempt to understand the nature of this coincidence.

Pythagorean thinking

Let us consider the following situation. A serious middle-aged scholar, the founder of a large scientific school, is writing a book on socio-political problems. In order to make his analysis more comprehendable, he constructs an imaginary model of a special city, which he calls Magnesia. He describes the structure of this city and people's functions within it. As was demonstrated by McClain that the scholar also included in the structure of the city certain numerical relations which would become the foundation of musical culture in later centuries. Why did Plato include these numbers in the structure of the city? Why are Plato's texts, in general, full of numerical ratios which, as was shown by McClain, have musical meanings? I believe that the answer lies in McClain's words:

Everything in Plato's mathematical world is held together by the three means—arithmetic, harmonic, and geometric (McClain, 1978, p. 11).

I would like to explain how I perceive this statement. Let us look at a triangle with sides which relate as 3 : 4 : 6. On one hand, we have a geometrical picture; on the other hand, this picture is related to certain numbers. We may manipulate them in our mind and find, for example, that the first side is two times smaller than the third one, and the second one is longer than the first, etc.

An educated Pythagorean's style of thinking was completely different from our own. For him a ratio of two numbers represented not only the ratio of the lengths of the two segments, but also the *symbol* of a musical interval. Glancing at this triangle, a Pythagorean could hear a corresponding musical interval in the same way as today's musician can hear musical sounds when looking at standard notation. Therefore, a Pythagorean, in addition to the geometrical picture of a triangle, also had its acoustic image in mind. These images integrating with each other generated a single 'steroscopic' image of a triangle. The geometric thinking of a Phythagorean had a musical dimension which is unknown to us.

When a Pythagorean used the phrase 'music of spheres' it was neither a metaphor nor an allegory: It was a 'subject's report' about real musical feelings while looking at geometrical figure. I would like to emphasize my main point: *a Pythagorean not only knew that certain numerical ratios correspond to musical sounds, but he also heard at least some of them while perceiving geometrical figures and numerical sequences.*

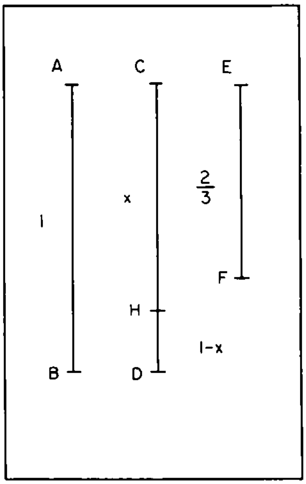

Fig. 2. Plato's harp. *Segment AB corresponds to a string of the length 1; segment EF to the string of the length 2/3. H is a mobile bridge; it can move in such a way that 1/2 < CH < 1; in addition, CH ≠ 2/3.*

While creating his ideal city, Plato could both see and hear it. He constructed this city not in the ordinary three-dimensional space, but rather in four-dimensional space with the fourth dimension corresponding to an ideal sound. It seems that Plato was the last great Pythagorean who could hear the 'singing' of geometrical figures. The ability to hear ratios of numbers and segments never went beyond his students. Later generations of Plato's readers could see only aesthetic metaphors in his numerical relations.

A reflexional model of a subject (Lefebvre, 1987) allows us to make an attempt of reconstructing the mathematical core of Plato's ontology, which Wheeler characterized as the integrity of psychology, mathematics, and harmony (Wheeler, 1982).

Reconstruction of a formal model

Of course, in Plato's time a formal model of reflexion could not be constructed. First, there was no formal concept of probability; second, algebraic technique necessary to construct such a model did not exist. However, one formal fragement of a relfexive model could have appeared during Plato's time and could have provided a formal frame for the model of *Universum*. I emphasize that in this reconstruction no thoughts regarding reflexion are ascribed to Plato. He could only use certain formal correlations similar to ours while giving them completely different interpretations.

Elaborating on McClain's monochord, I propose an abstract musical instrument: a board with three tight strings (Figure 2). Two of them (AB and CD) are of the same length; consider it equal to 1. The length of the third string, EF, is 2/3. String CD has a bridge H, which may move along the string in a way such that the length of CH is always greater than 1/2, smaller than 1, and never equal to 2/3. Strings AB and EF do not have bridges.

I will call this abstract construction *Plato's harp*. It is worth noting that Plato's harp is a harp in the same sense as Turing's machine is a machine: Both are abstract constructions.

Our next step will be to consider two intervals x and y: x is an interval between CH and AB, and y is an interval between CH and EF. Let us analyse two cases.

The first case

Let CH > EF. Introduce the following notations:

$$\frac{\mathrm{CH}}{\mathrm{AB}} = x \quad \text{and} \quad \frac{\mathrm{EF}}{\mathrm{CH}} = y.$$

Now, we will require that the following two equalities be true at the same time and find all possible values of x.

$$\frac{x}{1-x} = 1 + k_1, \quad k_1 = 1, 2, 3 \ldots,$$

$$\frac{y}{1-y} = 1 + k_2, \quad k_2 = 1, 2, 3 \ldots.$$

It is easy to see that $xy = 2/3$, because

$$xy = \frac{\mathrm{CH}}{\mathrm{AB}} \cdot \frac{\mathrm{EF}}{\mathrm{CH}} = \frac{\mathrm{EF}}{\mathrm{AB}} = \frac{2}{3}.$$

Therefore, the search for x is reduced to finding a solution for the following system of equations:

$$\left.\begin{aligned} \frac{x}{1-x} &= 1 + k_1 \\ \frac{y}{1-y} &= 1 + k_2 \\ xy &= \frac{2}{3} \end{aligned}\right\}, \qquad (1)$$

where k_1 and k_2 are natural numbers. Some easy transformations allows us to express k_1 as a function of k_2:

$$k_1 = \frac{k_2 + 5}{k_2 - 1}, \quad k_2 \geqslant 2.$$

Let us introduce a notation: $k_2 - 1 = m$; then

$$k_1 = 1 + \frac{6}{m}.$$

Since k_1 is an integer, m can take on only the values of the divisors of the number 6, that is, $m = 1, 2, 3, 6$. Thus,

$$k_1 = 2, 3, 4, 7.$$

By substituting k_1 in the first equation of the system (1) with these number we will obtain the following intervals:

$$3/4, \ 4/5, \ 5/6, \ 8/9.$$

The second case

Let CH < EF. We introduce the following notations:

$$\frac{\mathrm{CH}}{\mathrm{AB}} = x \quad \text{and} \quad \frac{\mathrm{CH}}{\mathrm{EF}} = y.$$

Now, we will requre that the following two equalities be true at the same time:

$$\frac{x}{1-x} = 1 + \frac{1}{k_1}, \quad k_1 = 1, 2, 3 \ldots,$$

$$\frac{y}{1-y} = 1 + k_2, \quad k_2 = 1, 2, 3, \ldots.$$

Similar to the first case, we find all of the possible values of x. It is easy to see that in this case $x/y = 2/3$, because

$$\frac{x}{y} = \frac{\mathrm{CH}}{\mathrm{AB}} : \frac{\mathrm{CH}}{\mathrm{EF}} = \frac{\mathrm{EF}}{\mathrm{AB}} = \frac{2}{3}.$$

Thus, a search for x is reduced to finding a solution of the following system of equations:

$$\left.\begin{aligned} \frac{x}{1-x} &= 1 + \frac{1}{k_1} \\ \frac{y}{1-y} &= 1 + k_2 \\ \frac{x}{y} &= \frac{2}{3} \end{aligned}\right\}, \tag{2}$$

where k_1 and k_2 are positive integers. By expressing k_1 as a function of k_2 we will obtain:

$$k_1 = \frac{k_2 + 4}{k_2 - 2}, \quad k_2 \geqslant 3.$$

If we introduce $m = k_2 - 2$, we again (!) come to the expression

$$k_1 = 1 + \frac{6}{m}.$$

After considerations similar to those in the first case, we obtain four possible intervals:

8/15, 5/9, 4/7, 3/5.

We can see now that the set of intervals found in analysis of the cases together with the interval 2/3 coincides with the set called *super-elite*. By adding their octave complements to them we obtain the Just Intonation set together with the two non-standard intervals 4/7 and 7/8 (see p. 74).

Conclusion

Plato's harp is a two-faced Janus. On one hand, it is a musical instrument; on the other hand, it is a geometrical object. This is a special abstract machine which allows Plato to hear geometrical proportions and to see musical intervals. It contains polarization of immobility (AB and EF) and mobility (CH). There is absolute harmony (the ratio of AB to EF is equal to a perfect fifth) as well as variable relations (an interval x depends on the location of H). There are natural numbers ($k_1 = 1, 2, 3, \ldots$) and a sequence of diminishing values ($k_1 = 1/1, 1/2, 1/3, \ldots$). Finally, we can see ideas of infinity ($k_1 \to \infty$) and of limit ($1/k \to 0$).

I believe that a thorough comparison of this construction with Plato's and Aristotle's original texts might allow us to find formal analogies to such unclear concepts as 'Unity', 'the Great and the Small', 'Unlimited and Limit', 'More and Less', 'Divided line', 'the Indefinite Dyad', and many other such concepts.

Acknowledgment

I am grateful to Victorina Lefebvre for her invaluable help in preparation of this paper and to Lev Levitin and Harvey Wheeler for their extremely important suggestions.

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Appendix. Plato's harp and a model of reflexion

Let us analyse how Plato's harp relates to the model of reflexion. When constructing a model of a subject perceiving musical intervals (Lefebvre, 1987) we represent him with the help of the following equation:

$$\left(\frac{1}{2}\right)^{z^x} = x. \tag{A1}$$

In this notation 'raising to power' is understood as the following operation:

$$p^q = p + (1 - q) - p(1 - q), \quad \text{and} \quad r_1^{r_2^{r_3}} = r_1^{(r_2^{r_3})}.$$

Deep feeling, D, and surface feeling, S are defined as:

$$D = 1 - z^x$$

$$S = 1 - x,$$

where $\frac{1}{2} \leqslant x < 1$.

Equation (A1) allows us to express z through x:

$$z = \frac{1 - x}{x}.$$

Thus, D can be represented as:

$$D = 2x - 1.$$

Now the ratio D/S looks

$$\frac{D}{S} = \frac{x}{1-x} - 1.$$

We require that the ratio of feelings be an integer. Two chains of equalities correspond to this requirement:

$$\frac{D}{S} = \frac{x}{1-x} - 1 = k_1; \quad k_1 = 1, 2, 3, \ldots,$$

$$\frac{D}{S} = \frac{x}{1-x} - 1 = \frac{1}{k_1}; \quad k_1 = 1, 2, 3, \ldots.$$

Let us add 1 to each element of these chains and obtain new chains:

$$\frac{D+S}{S} = \frac{x}{1-x} = 1 + k_1, \tag{A2.1}$$

$$\frac{D+S}{S} = \frac{x}{1-x} = 1 + \frac{1}{k_1}. \tag{A2.2}$$

We see that the equality (A2.1) corresponds to the first equation in the system (1), and the equality (A2.2) corresponds to the first equation in the system (2). Now we can understand the sense of the expression $x/(1-x)$ in the formal model of reflexion. *This is a ratio of a subject's summary feeling* $(D + S)$ *to his superficial feeling* (S) *when perceiving interval* x.

The second equations in the systems (1) and (2) are interpreted similarly: the expression $y/(1-y)$ is a ratio of a subject's summary feeling to his surface feeling when he is perceiving an interval y. In the first case $y = 2/3x$, and in the second case $y = 3x/2$.

Correlation between the intervals x and y are given with the help of the third equations in the two systems (1) and (2).